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A piecewise linear proof that the singular norm is the Thurston norm

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Abstract

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A natural generalization of the Thurston norm is the singular norm. We prove that on a compact oriented 3-manifold, the singular norm is the Thurston norm. This theorem, conjectured by William Thurston, has been proven by David Gabai using sutured manifold hierarchies, foliation theory and minimal surface techniques. Our proof is combinatorial and, except for the existence of sutured manifold hierarchies, mostly self-contained.

Keywords: Sutured manifold; 3-manifold; Thurston norm.

AMS (MOS) Subj. Class.: 57M50, 55N99.

1. Introduction

Definition 1.1. Let M be a compact oriented 3-manifold, and let N be a subsurface of ∂M. The *Thurston norm* x on $H_2(M, N)$ is defined by

$$x(\tau) = \min\{\chi_-(T) \colon T \text{ is a properly embedded oriented surface with } \partial T \subset N \text{ and } [\mathrm{T}] = \tau\},$$

where

$$\chi_-(T) = \max\{0, -\chi(T)\}$$

for connected surfaces T, and in general,

$$\chi_-(T) = \sum \chi_-(T_i),$$

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where the sum is over all connected components T_i of T. Alternatively, one may define the *singular norm* x_s on $H_2(M, N)$ by

$$x_s(\tau) = \min\{n^{-1}\chi_-(T): n \text{ is a positive integer, } T \text{ is a closed oriented surface and there exists a map } f: (T, \partial T) \to (M, N) \text{ with } f_*[T] = n\tau\}.$$

The aim of this paper is to prove the following.

Theorem 1.2. *If M is a compact oriented 3-manifold and N is a subsurface of ∂M, then $x_s = 2$.*

This theorem in the case where $N = \partial M$ has been proven by Gabai in [1] using sutured manifold hierarchies, foliation theory and minimal surface techniques. The proof here is combinatorial and, except for the existence of sutured manifold hierarchies, mostly self-contained. The chief calculation is this: out of all ways to inscribe an n-gon in D^2, an embedding maximizes the sum of the interior angles. See Fig. 1.

2. Review of sutured manifolds

For details on sutured manifolds see [4].

Definitions 2.1. A pair (M, γ) is a *sutured manifold* if M is an oriented compact 3-manifold, γ is an oriented compact 1-submanifold of ∂M, and R_+ and R_- are oriented 2-submanifolds of ∂M with $\partial M = R_+ \cup (-R_-)$ and $\partial R_+ = \partial R_- = \gamma$. In particular, if $\partial M = \emptyset$, then $(M, \emptyset)$ is a sutured manifold. A component of γ is called a *suture*, and a regular neighborhood of γ in ∂M is denoted $A(\gamma)$.

Given a properly embedded oriented surface $(S, \partial S) \subset (M, \partial M)$ which intersects each torus boundary component of R_+ or R_- in coherently oriented parallel circles and which intersects $A(\gamma)$ in essential arcs, there is a natural way to define a sutured structure $(\tilde{M}, \tilde{\gamma})$ on the 3-manifold $\tilde{M}$ obtained by cutting M along S. See Figs. 2(a) and (b).

The process of cutting M along S and obtaining $(\tilde{M}, \tilde{\gamma})$ is called a *sutured decomposition* and is denoted $(M, \gamma) - S \to (\tilde{M}, \tilde{\gamma})$.

Fig. 1. In the first 4-gon the interior angle sum is not maximal, but the second 4-gon is embedded and it maximizes the interior angle sum.

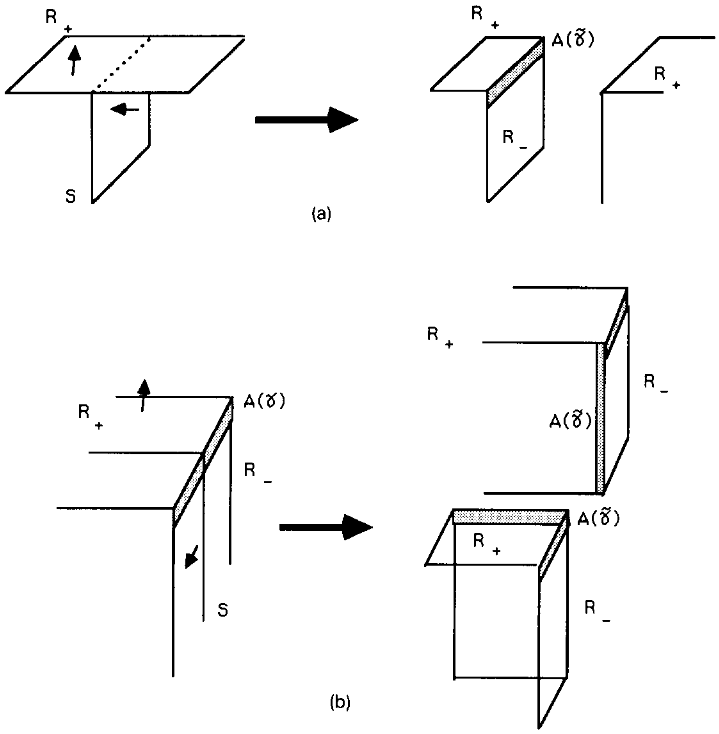

Fig. 2. (a) On the left is the local picture in M where S does not intersect γ; arrows denote normal direction. On the right is the result in $\tilde{M}$ after cutting along S. (b) On the left is the local picture in M where S does intersect γ; again arrows denote normal direction. On the right is the result in $\tilde{M}$ after cutting along S.

A surface $(S, \partial S) \subset (M, \partial M)$ is *taut* if S is incompressible and norm-minimizing, i.e., $\chi_-(S) = x([S, \partial S])$ where $[S, \partial S] \in H_2(M, \eta(\partial S))$ and $\eta(\partial S)$ denotes a regular neighborhood of ∂S.

A sutured manifold (M, γ) is *taut* if M is irreducible and both R_+ and R_- are taut. A sutured decomposition $(M, \gamma) - S \to (\tilde{M}, \tilde{\gamma})$ is *taut* if both (M, γ) and $(\tilde{M}, \tilde{\gamma})$ are taut. A *sutured hierarchy* for M is a sequence

$$(M, \gamma) - S_1 \to (M_1, \gamma_1) - S_2 \to \cdots \to (M_{n-1}, \gamma_{n-1}) - S_n \to (M_n, \gamma_n)$$

of taut sutured decompositions such that M_n is a disjoint union of 3-balls each of which contains exactly one suture. In [4] this is called a *taut* sutured manifold hierarchy, but since we will use no other kind of hierarchy, we drop the word taut.

We state the following theorem without proof (see [4, §4.19]).

Theorem 2.2. *Given a taut sutured manifold* (M, γ) *and a nontrivial element* $\tau \in H_2(M, \partial M)$, *there exists a sutured hierarchy*

$$(M, \gamma) - S_1 \to (M_1, \gamma_1) - S_2 \to \cdots \to (M_{n-1}, \gamma_{n-1}) - S_n \to (M_n, \gamma_n)$$

for M *with* $[S_1, \partial S_1] = \tau$ *in* $H_2(M, \partial M)$.

Remark 2.3. In Lemma 6.6 we use the additional fact that we may require for each j that no component of ∂S_j ever bounds a disk in R_+ or R_-.

3. A lemma and the outline of the proof of Theorem 1.2

Clearly $x_s \leqslant x$; hence it remains to prove $x \leqslant x_s$.

Lemma 3.1. *It suffices to prove Theorem* 1.2 *in the case M is an irreducible closed oriented 3-manifold.*

Proof. Suppose we know Theorem 1.2 is true for closed irreducible 3-manifolds. We may as well take M connected.

Case 1: $\partial M \neq \emptyset$, $N = \emptyset$, *and M is irreducible and ∂-irreducible.*

Let $D(M) = M_1 \cup M_2$ denote the double of M along ∂M, where M_1 and M_2 are copies of M. Since M is irreducible and ∂-irreducible, $D(M)$ is irreducible. Let i: $M_1 \to D(M)$ be the inclusion and d: $D(M) \to D(M)$ the doubling involution.

For a given $\tau \in H_2(M_1)$, let R be a norm-minimizing embedded surface in $D(M)$ representing $i_*(\tau)$ in $H_2(D(M))$. Since $D(M)$ is irreducible, we may assume R contains no sphere components. Put R in general position with respect to ∂M_1 $(= \partial M_2)$, and isotope to minimize $|R \cap \partial M_1|$, the number of components of $R \cap \partial M_1$. Let $R_i = R \cap M_i$, $i = 1, 2$. Since M is irreducible and ∂-irreducible, we may assume no component of R_i is a disk. The image of $i_*(\tau) = [R]$ under the excision isomorphism $H_2(D(M), M_1) \cong H_2(M_2, \partial M_2)$ is trivial. Hence $[R_2]$ is trivial in $H_2(M_2, \partial M_2)$ so the fundamental class of $R_2^* = d(-R_2)$ is trivial in $H_2(M_1, \partial M_1)$. It follows that the closed singular surface $\tilde{R} = R_1 \cup R_2^*$ also represents $i_*(\tau)$ in $H_2(D(M))$. Since $H_3(D(M)) \to H_3(D(M), M_1)$ is an isomorphism, $i_* : H_2(M_1) \to H_2(D(M))$ is injective and $[\tilde{R}] = \tau$ in $H_2(M_1)$.

Push $\tilde{R} \cap \partial M_1$ into M_1 so that the inclusion of $\tilde{R}$ into M_1 is proper. Notice that $\tilde{R}$ has only closed double curves of intersection.

Claim 1. *After perhaps an isotopy of R_2^* or R_1* rel $\partial R_1 = \partial R_2^*$, *each closed double curve is essential in both R_1 and R_2^*.*

Proof. Suppose some closed double curve is inessential in either R_1 or R_2^*, say in R_1. Find an innermost such curve λ. Since R was Thurston norm-minimizing and neither R_1 nor R_2 contains disks, R_2^* is incompressible in M_1, so λ is inessential in R_2^* as well. Using the irreducibility of M_1 we may isotope the disk in R_2^* which λ bounds parallel to that bounded by λ in R_1, eliminating the double curve λ (and perhaps others). After a sufficient number of such moves we can ensure that each closed curve of intersection is essential in both R_1 and R_2^*.

Let $T \subset M_1$ be the closed surface obtained by taking the double curve sum of $\tilde{R}$ along its double curves. Then $[T] = \tau$, and $\chi(T) = \chi(R_1) + \chi(R_2) = \chi(R)$.

Claim 2. $\chi_-(T)=\chi_-(R)$.

Proof. It suffices to show that T has no sphere components. Since R contains no sphere components, no sphere component of T lies entirely in R_1 or entirely in R_2^*. Hence each such component is made up of bits of R_1 glued to bits of R_2^*, with the gluing along a closed 1-manifold Λ. Each component of Λ is either a double curve or a component of $R \cap \partial M_1$.

Suppose λ is a component of Λ, innermost in a sphere of T. Then λ is inessential in either R_1 or R_2^*. But we have eliminated all inessential closed double curves, so λ must be a component of $\partial R_1 = \partial R_2^*$. But then the disk it bounds in T would be a disk component of either R_1 or R_2^*, whereas neither contains disk components. The contradiction proves the claim.

To complete the proof in this case, consider a singular map $f: S \to M$ with $f_*[S]=n\tau$. Then apply Theorem 1.2 to the class $i_*(\tau)$ in $H_2(D(M))$ to obtain $X_-(R) \leqslant n^{-1}\chi_-(S)$. But then $\chi_-(T)=\chi_-(R)\leqslant n^{-1}\chi_-(S)$, proving Theorem 1.2 for M.

Case 2: M is irreducible, $N \neq \emptyset$, each curve of ∂N is essential in ∂M, and

$$\textit{if } E \textit{ is any } \partial\textit{-reducing disk for } M, |\partial E \cap \partial N| \geqslant 4. \qquad (*)$$

Since each curve of ∂N is essential in ∂M, condition $(*)$ ensures that both N and $\partial M - N$ are incompressible in M.

Let $D(M)=M_1 \cup M_2$ denote the double of M along N, where M_1 and M_2 are copies of M. Since M is irreducible and N is incompressible, $D(M)$ is irreducible.

Claim. *$D(M)$ is also ∂-irreducible.*

Proof. Suppose $(E, \partial E) \subset (D(M), \partial D(M))$ is a disk in general position with respect to N, isotoped to minimize $|N \cap E|$. Since N is incompressible and M is irreducible, there are no closed components of $N \cap E$. In particular, since $|N \cap E|$ is minimized so is $|\partial N \cap \partial E| = 2|N \cap E|$. Consider an outermost arc of $N \cap E$ in E. It cuts off a disk F in M_1, say, whose boundary consists of an arc in $\partial M_1 - N$ and an arc in N. In particular, $|\partial F \cap \partial N| = 2$. By $(*)$, ∂F then bounds a disk F' in ∂M_1 intersecting ∂N in a single arc. Half this disk provides an isotopy in $D(M)$ of the arc $\partial F - N$ to a subarc of ∂N, lowering $|\partial E \cap \partial N|$ by two. We conclude that $N \cap E = \emptyset$, so E lies entirely in M_1. Since $\partial M - N$ is incompressible, E is parallel in M_1 to a subdisk of $\partial M_1 - N \subset \partial D(M)$, proving the claim.

Let $d: D(M) \to D(M)$ be the doubling involution, so $d(M_1) = -M_2$. Let $\tau \in H_2(M_1, N)$, and let R be a norm-minimizing embedded surface in $D(M)$ representing $\tau - d_*(\tau) \in H_2(D(M))$. We may suppose that R is in general position with respect to N, and delete any component which is a sphere. Let $R_i = R \cap M_i$, $i=1$, 2. Since N is incompressible, any disk component of R_i is parallel to a disk in N. Hence all disk components of R_i may be eliminated by an isotopy of R which does not increase $|R \cap N|$. Then $\chi_-(R) = -\chi(R) = -(\chi(R_1)+\chi(R_2)) = \chi_-(R_1) + \chi_-(R_2)$. With no loss assume $\chi_-(R_1) \leqslant \chi_-(R_2)$, so $2\chi_-(R_1) \leqslant \chi_-(R)$.

Suppose $f:(S_1, \partial S_1) \to (M_1, N)$ is a singular surface representing $n\tau$. Let $S = D(S_1)$ be the double of S_1, and $d(f): S \to D(M)$ be the naturally induced singular map. Then $d(f)_*([S])$ is a closed singular surface representing $n(\tau - d_*(\tau))$ in $D(M)$. Since $\chi(S) = 2\chi(S_1)$ and sphere components of S come precisely from doubling disk and sphere components of S_1, it follows that $\chi_-(S) = 2\chi_-(S_1)$. Hence we have $2n^{-1}\chi_-(S_1) = n^{-1}\chi_-(S) \geqslant \chi_-(R) \geqslant 2\chi_-(R_1)$. Finally, since $n[R] = d(f)_*([S])$ in $H_2(D(M))$, their images coincide under $H_2(D(M)) \to H_2(D(M), M_2) \cong H_2(M_1, N)$. Hence $[R_1] = \tau$ in $H_2(M_1, N)$, verifying Theorem 1.2 in this case.

Case 3: M is irreducible, and again

$$\text{if } E \text{ is any } \partial\text{-reducing disk for } M, |\partial E \cap \partial N| \geqslant 4. \qquad (*)$$

Following Cases 1 and 2, the proof will be by induction on the number of inessential curves of ∂N. Clearly $(*)$ remains true after removing inessential curves of ∂N. Let D be a disk bounded by an innermost inessential curve in ∂N.

If D lies in N, let $N' = N - D$. Since $H_2(M, N) \cong H_2(M, N')$, the proof follows by induction.

If D lies in $M - N$, let $N' = N \cup D$. Then the kernel of the epimorphism $j: H_2(M, N) \to H_2(M, N')$ is generated by the fundamental class of D. Let τ be a class in $H_2(M, N)$ and $f:(S, \partial S) \to (M, N)$ be a singular surface representing $n\tau$. By induction, there is an embedded surface $(T, \partial T) \subset (M, N')$ with $[T] = j(\tau)$ and $\chi_-(T) \leqslant n^{-1}\chi_-(S)$. By general position we may take $\partial T \subset N$ and consider $[T]$ in $H_2(M, N)$. Then $j([T]) = j(\tau)$. After perhaps adding to T some number of copies of D, which has no effect on its norm, we have $[T] = \tau$ in $H_2(M, N)$.

Case 4: The general case.

Call a ∂-reducing disk for M which intersects ∂N in no more than two points *special*. There is a collection J of disjoint 2-spheres and special disks such that the result M' of cutting M along J and capping off the 2-sphere boundary components is irreducible and satisfies $(*)$ of Case 3. Let $N' \subset \partial M'$ be the union of N and the disks in $\partial M'$ created by cutting M open along the disks of J.

Let $\tau \in H_2(M, N)$ and $f:(S, \partial S) \to (M, N)$ be a singular map representing $n\tau$. The boundary of any special disk is either disjoint from N or intersects it in a single component. Hence (singular) compressions and ∂-compressions of the image of S change it to a map $f':(S', \partial S') \to (M, N)$ with $\chi_-(S') \leqslant \chi_-(S)$, $f'_*([S']) = n\tau$, and $f'(S') \cap J = \emptyset$. Then we may regard f' as a map into (M', N').

Let j be the epimorphism $j: H_2(M, N) \to H_2(M, N \cup J) \cong H_2(M', N')$. By Case 3 there is an incompressible oriented surface $(T, \partial T) \subset (M', N')$ with $\chi_-(T) \leqslant n^{-1}\chi_-(S')$ and $[T] = j(\tau)$ in $H_2(M', N')$. By general position we may take T to be disjoint from the balls in M' that the spheres of J bound and, exploiting the incompressibility of T, ∂T to be disjoint from the disks $N' - N$. Hence $T \subset M - J$. Viewing $[T]$ then as a class in $H_2(M, N)$ we have $j([T]) = j(\tau)$ in $H_2(M', N')$. But $\ker(j)$ is generated by the fundamental classes of spheres in J and of disks in J whose boundary lies entirely in N. So, after perhaps altering T

by adding copies of components of J, all of which have trivial Thurston norm, $[T]=\tau$ and $\chi_-(T)\leqslant n^{-1}\chi_-(S')\leqslant n^{-1}\chi_-(S)$. □

3.1. Outline of the proof of Theorem 1.2

Suppose then that M is irreducible closed and oriented, and let Σ be the set of all maps of closed oriented surfaces into M. To prove Theorem 1.2, we will construct for any given hierarchy for M a function y (which depends on the hierarchy) from Σ to the integers satisfying the following properties:

(i) If $f_i: T_i \to M$ is an element of Σ for $i=1, 2$, and if $f_{1*}[T_1]=f_{2*}[T_2]$, then $y(f_1)=y(f_2)$.

(ii) If $f: T\to M$ is in Σ, n is an integer, and nf is the map on n disjoint copies of T defined by $(nf)(t)=f(t)$, then $y(nf)=ny(f)$.

(iii) If $f: T\to M$ is in Σ, then $y(f)\leqslant\chi_-(T)$.

We will also show:

(iv) Given a nonzero element τ of $H_2(M)$, there exists a hierarchy for M and an embedding $g: T\to M$ of a surface T so that $y(g)=\chi_-(T)$ and $g_*[T]=\tau$.

Aside. The proof in fact shows that there exists a function Y from the set of hierarchies on M to $H^2(M)$ so that

(1) $Y(H)(\tau)|\leqslant x(\tau)$ for all hierarchies H and all $\tau\in H_2(M)$, and

(2) for every nonzero $\tau\in H_2(M)$, there exists H with $Y(H)(\tau)=x(\tau)$.

In fact, $Y(H)$ is the Euler class of the 2-plane bundle tangent to the foliation determined by the hierarchy H. However, we will use nothing about foliations in this proof.

Once y is defined and (i)–(iv) are proved, the following easy argument finishes the proof of Theorem 1.2:

Let τ be a nontrivial element of $H_2(M)$. Then by (iv), there exists a hierarchy and an embedding $g: T\to M$ with $y(g)=\chi_-(T)$ and $g_*[T]=\tau$. Let $f: S\to M$ be an element of Σ and n a positive integer with $f_*[S]=n\tau$. Then

$$\chi_-(T)=y(g)=n^{-1}y(ng)=n^{-1}y(f)\leqslant n^{-1}\chi_-(S).$$

Since $\chi_-(T)\leqslant n^{-1}\chi_-(S)$ for all $f: S\to M$ in Σ, $\chi_-(T)\leqslant x_s(\tau)$, and hence $x(\tau)\leqslant x_s(\tau)$.

4. Definition of y

We will work in the pl category: assume all 1-, 2-, and 3-manifolds are pl, all submanifolds are pl-submanifolds, and all maps are pl. Let $I=[-1,1]$.

Let (M, γ) be a taut sutured 3-manifold, and fix a sutured hierarchy

$$(M_0, \gamma_0)-S_1\to(M_1,\gamma_1)-S_2\to\cdots\to(M_{n-1},\gamma_{n-1})-S_n \to(M_n,\gamma_n)$$

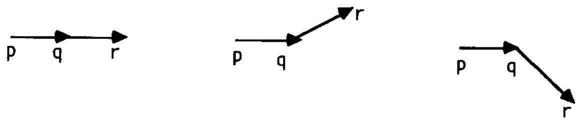

Fig. 3. In the first configuration, ext$(pqr) = 0$. In the second, ext$(pqr) = \pi/4$ and int$(pqr) = 3\pi/4$. In the last, ext$(pqr) = -\pi/4$ and int$(pqr) = -3\pi/4$.

for (M, γ), where $(M_0, \gamma_0) = (M, \gamma)$ and M_n is the disjoint union of 3-balls with one suture in each. Even when $\partial M \neq \emptyset$, we will define y on a class of maps $f: T \to M$ of surfaces T to M; however, only when M is closed will y satisfy properties (i)–(iv) of Section 3.1.

Notation 4.1. (i) *Angles*: Let D^2 be the standard disk $I \times I$ (so that angle measure in D^2 is defined). Given three distinct points p, q, and r in D^2, let ext$(pqr) \in (-\pi, \pi)$ denote the radian measure of the exterior angle at q of the path which is a straight arc from p to q followed by a straight arc from q to r. When ext$(pqr) \neq 0$, let int(qr) denote the unique number in $(-\pi, \pi)$ with

$$\mathrm{int}(pqr) + \mathrm{ext}(pqr) = \pi \text{ modulo } 2\pi.$$

See Fig. 3.

(ii) *Definition of $H: M \to M$ and subsets $\tilde{C}$, $\tilde{A}$, C, $A \subset M$*: Frequently $\eta(\cdot)$ will denote a regular neighborhood of $\cdot$.

For $j = 1, 2, \ldots, n$, we have

$$M_j = M_{j-1} - \mathrm{Int}(\eta(S_j))$$

for some regular neighborhood $\eta(S_j)$ in M_{j-1} of S_j; in this way we regard each M_j as a subset of M. Let $i: M_n \to M$ be inclusion, and define $\tilde{C} = i(\partial M_n)$ and $\tilde{A} = i(A(\gamma_n))$.

In M_{j-1}, $\eta(S_j)$ is topologically the product $S_j \times I$; fix an identification $\eta(S_j) = S_j \times I$. Let $H: M \to M$ be the result of the following sequence of shrinkings: first shrink the I-fibres of $\eta(S_n) = S_n \times I$ to $S_n \times \{0\}$, then shrink the I-fibers of $\eta(S_{n-1}) = S_{n-1} \times I$ to $S_{n-1} \times \{0\}$, etc., ending with shrinking the I-fibers of $\eta(S_1)$. Put $C = H(\tilde{C})$ and $A = H(\tilde{A})$. See the schematic Figs. 4(a) and (b).

(iii) *Definition of g, Γ, and the c_k*: Suppose that M_n has m components. Let h be a homeomorphism from m disjoint copies of $D^2 \times I$ to M_n so that on each copy of $D^2 \times I$, h takes $\partial D^2 \times \{0\}$ to a suture and $\partial D^2 \times I$ to a component of $A(\gamma_n)$. Define $g = ih$, where $i: M_n \to M$ is inclusion.

Let Γ be the set of all proper maps $f: T \to M$ of an oriented surface T into M such that

(a) f is in general position with respect to $\tilde{C}$, and for each j, $(f(T) \cap \eta(S_j) = (f(T) \cap S_j) \times I$ in the fixed product structure of $\eta(S_j) = S_j \times I$, and

(b) f respects the fibers of $\tilde{A}$, i.e., $g^{-1}(f(T) \cap \tilde{A})$ consists of arcs of the form $\{x\} \times I$.

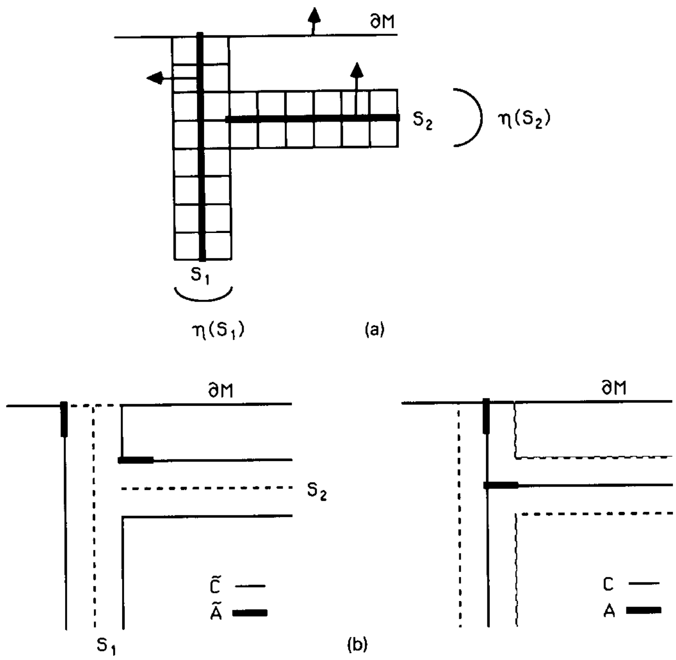

Fig. 4. (a) Here $\eta(S_j) = S_j \times I$ is shown as a subset of M. (b) The figure on the left shows $\tilde{C}$ and $\tilde{A}$ in M. The figure on the right shows C and A in M.

Claim. *Any element of Σ is homotopic to an element of Γ.*

Proof. Let $S = \bigcup\{S_i : 1 \leqslant i \leqslant n\}$. By general position we may assume that $H \mid \partial S : \partial S \to C$ is an embedding, except perhaps at a finite set P of double points. Homotope T so that it is in general position with respect to $C = H(S)$, $H(\partial S)$, and P. Since T is then disjoint from P, we may align $T \cap C$ so that near $H(\partial S)$ it runs along fibers $H(\{x\} \times I)$ for $x \in \partial S$. Then $H^{-1}(T)$ is surface homotopic to T in M and satisfies the required conditions.

If $f : T \to M$ is an element of Γ, then $T' = f^{-1}(i(M_n))$ is a codimension-0 submanifold of T with boundary components, say, $\delta_1, \delta_2, \ldots, \delta_p$ which inherit an orientation from T'. For $k = 1, 2, \ldots, p$, let $s_k : S^1 \to \delta_k$ be an orientation preserving homeomorphism, and define oriented curves $c_k : S^1 \to \partial(D^2 \times I)$ by $c_k = g^{-1} g s_k$.

Definition 4.2 (of y). Let $\rho : D^2 \times I \to D^2$ be projection to the first coordinate. If $c : S^1 \to \partial(D^2 \times I)$ is any pl general position map with $c(S^1)$ meeting $\partial D^2 \times I$ in fibers of the form $\{x\} \times I$, then $\rho c(S^1)$ is a finite collection of straight arcs. If p, q and r are distinct points in D^2 and if while traversing c in the positive direction, pq and qr are successive straight arcs in $\rho c(S^1)$, let $\theta(pqr) = \operatorname{int}(pqr)$ if $q \in \partial D^2$ and let $\theta(pqr) = -\operatorname{ext}(pqr)$ if $q \in \operatorname{Int}(D^2)$. Define $y(c) = \Sigma\theta(pqr)/2\pi$, where the sum is over all such successive arcs pq and qr in $\rho c(S^1)$.

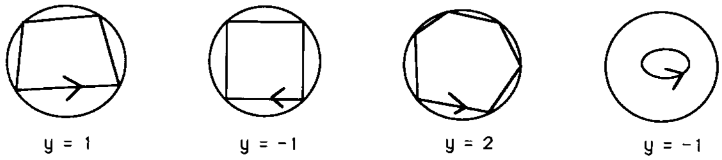

Fig. 5.

We define y on elements $f:T\to M$ of Γ. If $f(T)\cap\tilde{C}=\emptyset$, put $y(f)=0$; otherwise, put

$$y(f)=\sum y(c_k),$$

where the sum runs over $k=1,2,\ldots,p$.

If $\partial M=\emptyset$, then given an arbitrary element $f':T\to M$ of Σ, define $y(f')=y(f)$, where f is any element of Γ homotopic to f'. That y is well defined will follow from the fact, proved in Proposition 5.1, that $y(f_1)=y(f_2)$ whenever $\partial M=\emptyset$ and f_1 and f_2 are homotopic elements of Γ.

Notes 4.3. (i) If ρc is embedded, then

$$y(c)=\epsilon(\xi-2)/2,$$

where ξ is the number of intersections of c and $\partial D^2\times\{0\}$, and $\epsilon=\pm1$ is positive if and only if c travels counterclockwise. See Fig. 5.

Notice that for the purpose of computing $y(c)$, it does not matter what portions of the curve c are "on top" in $D^2\times\{1\}$ or "underneath" in $D^2\times\{-1\}$ because $y(c)$ depends only on the projected curve $\rho c:S^1\to D^2$.

(ii) If c is an embedding and if ρc either intersects ∂D^2 or travels counterclockwise, then

$$y(c)\leqslant(\xi-2)/2,$$

where again ξ is the number of intersections of c and $\partial D^2\times\{0\}$. See Fig. 6.

(iii) If ρc_k is embedded and travels counterclockwise for each k, then applying note (i) above to each c_k shows that

$$y(f)=\tfrac{1}{2}\sum(\xi_k-2),$$

where the sum runs over $k=1,2,\ldots,p$, and ξ_k is the number of points in $f(\delta_k)\cap i(\gamma_n)$.

Fig. 6.

(iv) If for each k, c_k is an embedding and ρc_k either intersects ∂D^2 or travels counterclockwise, then

$$y(f) \leqslant \tfrac{1}{2} \sum (\xi_k - 2),$$

where the sum runs over $k = 1, 2, \ldots, p$, and ξ_k is the number of points in $f(\delta_k) \cap i(\gamma_n)$.

(v) Suppose $\partial M = \emptyset$. Then once it is established in Proposition 5.1 that y is well defined on all of Σ, it is clear that $y(nf) = ny(f)$ for all $f \in \Sigma$ and integers n. This is property (ii) in Section 3.1.

5. y is a cocycle

Proposition 5.1. *Suppose M is a closed oriented 3-manifold with a given sutured manifold hierarchy. Let $y : \Gamma \to Z$ be the function defined above. If $f_i : T \to M (i = 1, 2)$ are homotopic elements of Γ, then $y(f_1) = y(f_2)$.*

Proof. Let $f : T \to M$ be an element of Γ. Note that $y(f)$ is determined by $f(T) \cap C$. Now, any arc α in $f(T) \cap C$ which misses A corresponds to two arcs in $\tilde{C}$—one of which, say α_+, satisfies $g^{-1}(\alpha_+) \subset D^2 \times \{1\}$ and the other, say α_-, satisfies $g^{-1}(\alpha_-) \subset D^2 \times \{-1\}$. In general a proper homotopy of α missing A changes the contribution to y of each $\rho(g^{-1}(\alpha_\pm))$ by an integer as loops appear and disappear. But $\rho(g^{-1}(\alpha_+))$ and $\rho(g^{-1}(\alpha_-))$ have opposite orientations so the change in their net contribution to y is zero.

A similar argument may be applied to some other elementary homotopies of f that leave $f(T) \cap A$ unchanged. For example, if a trivial circle of $f(T) \cap C$ is introduced by pushing a small disk of T through C away from A, the circle shows up in $D^2 \times I$ as two circles, one in each of $D^2 \times \{1\}$ and $D^2 \times \{-1\}$, having opposite orientation. So the net change in y is zero. Similarly, imagine what happens when a "saddle" of T passes through C away from A. This will change a pair of arcs, the first with endpoints a and b and the second with endpoints c and d to a pair of arcs, one with endpoints a and c and the other with endpoints b and d. There will be a corresponding change on both the ends $D^2 \times \{1\}$ and $D^2 \times \{-1\}$, but the arcs will have opposite orientation on each end, so there is no change in y. We now show that these moves are representative.

Any element of Γ is in general position and so is an immersion except at possible branch points; such maps are called *branched immersions*. Hall proves in [2] that two homotopic branched immersions have a homotopy between them in which each level is a branched immersion. Since the branch set is dimension zero at each level, any homotopy between elements of Γ may be replaced by a sequence of homotopies, each of which is one of the following three types: (1) the homotopy does not change intersection with C and is one in which branch points are created, eliminated, coalesced or divided, (2) the homotopy moves a branch point but does not change intersection with A, and (3) the homotopy is fixed outside some 3-ball $B \subset M$ and in B it is an immersion at each level. Since y is unchanged under homotopies of type (1) and (2), we restrict our attention to those of type (3).

As Hass and Hughes pointed out in [3], by applying general position arguments to homotopies between general position immersions, it may be assumed that each level is in general position (with respect to itself) except for the following moves and their inverses: (i) part of a sheet passes through another sheet, creating a double curve; (ii) a saddle passes through a sheet; (iii) a double curve passes through a sheet, creating two triple points; and (iv) a triple point passes through a sheet. Since we can require all these moves to happen off C, we consider only homotopies of type (3) in which at each level the immersion stays in general position with respect to itself. Hence we are interested in the effect on y of a sheet, saddle, double curve, or triple point passing through C.

Consider the following special case of a sheet passing through C. For any positive number ϵ, define a wedge in $D^2 \times I$ by

$$W = \{(x, y) \in D^2 : 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant \epsilon x\} \times I.$$

Let g' be the restriction of g to one copy of $D^2 \times I$ and define $G = Hg'$. Suppose B is a 3-ball in M containing $G(W)$ such that $B \cap A \subset G(W)$. Let T be a closed surface and let $F_t : T \to M$ be a homotopy such that F is fixed off a disk $U \subset T$ and $F_t(U) \subset B$ for all t. Further suppose $F_t(U)$ is always "vertical" in $G(W)$, i.e., $F_t(U) \cap G(W) = G(\alpha_t \times I)$ for some subset $\alpha_t \subset D^2$, where α_t is either an arc, a point, or the null set. Figure 7 may be helpful.

Then the contribution to y of $G(\partial(\alpha_t \times I))$ is zero for all t. Moreover, y is constant off $G(W)$; hence $y(F_0) = y(F_1)$.

Now, we may require sheets, saddles, and triple points passing through C to miss A if we also allow the above simple case of a sheet passing through C. (Note that we can insist we are dealing with only one wedge at a time because $Hi(\gamma_n)$ has no self-intersections.) To show y is unchanged when a double curve is pulled through A, one can again assume the situation is sufficiently simple and conclude that in fact $y(c_k)$ is constant for each k. Alternatively, one may use a variation of the argument in Lemma 6.4. □

Corollary 5.2. *When M is closed, y is a cocycle.*

Proof. We use the fact that if $f_*[T] = f'_*[T']$, then the two maps are cobordant. Hence there is a sequence of maps $f_i : T_i \to M$, $i = 1, \ldots, p$, with $f_1 = f$, $f_p = f'$, and f_i and f_{i+1} related by one of the following or its inverse:

(1) $T_i = T_{i+1}$ and f_{i+1} are homotopic.

(2) T_{i+1} is built from T_i by deleting two open discs and attaching an annulus; $f_{i+1} = f_i$ off the open disks and f_{i+1} on the annulus is the restriction to $\partial D^2 \times I$ of a map $D^2 \times I \to M$.

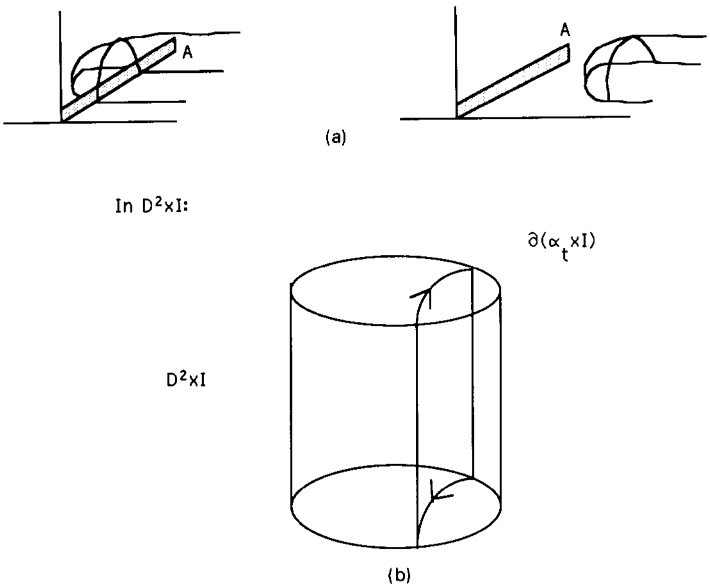

Fig. 7.Above on the left is a picture in M of $F_0(U)$ and on the right is $F_1(U)$. Note that below the horizontal plane in the first diagram, the movement of T changes nothing, hence is irrelevant to the calculation of y. This is true even if some of A lies below the plane, but perpendicular to the part of A shown.

It was already shown that y is invariant under (1). For (2), note that we may assume that the map on $D^2 \times I$ misses A altogether and so has no effect on y. $\square$

When M is not closed, y need not be invariant under homotopy; however, y is still an invariant in a weaker sense, motivating the following.

Definition 5.3. Let (M, γ) be a taut sutured manifold. A homotopy of a proper map $f: T \to M$ is *admissible* if it fixes $f(T) \cap A(\gamma)$ and restricts to a regular homotopy on ∂T.

Remark 5.4. By arguments similar to those above, $y(f_1) = y(f_2)$ even if $\partial M \neq \emptyset$, as long as f_1 and f_2 are elements of Γ related by an admissible homotopy.

6. Proof of property (iii) in Section 3.1: $y(f) \leqslant \chi_-(T)$ whenever M is closed and $f: T \to M$ is in Σ

The following definition and lemma are motivated by [4, §7.4].

Definition 6.1. Let (M, γ) be a sutured manifold, and suppose $f:(T, \partial T) \to (M, \partial M)$ is a proper branched immersion with $f(T) \cap \gamma$ consisting of a finite number ν of points, none of which is a double point of f. Define $I(f)$, the *index of f* in (M, γ) by

$$I(f) = \nu - 2\chi(T).$$

Suppose that S is a properly embedded surface in M, $\eta(S)$ is a regular neighborhood of S, $\tilde{M} = M = \operatorname{Int}(\eta(S))$, and $(M, \gamma) - S \to (\tilde{M}, \tilde{\gamma})$ is a sutured manifold decomposition. Let $f: T \to M$ be a branched immersion in general position with respect to S. Identify $\eta(S)$ with $S \times I$ and perform a homotopy on f so that $f(T) \cap \eta(S) = (f(T) \cap S) \times \mathrm{I}$. Let $\eta(\partial S)$ be a regular neighborhood of $\eta(S) \cap \partial M$ in $\eta(S)$ so that $A(\tilde{\gamma}) \cap (S_+ \cup S_-) \subset \eta(\partial S)$, where in $\eta(S) = S \times I$, $S_+ = S \times \{1\}$ and $S_- = S \times \{-1\}$. Identify $\eta(\partial S)$ with $(\eta(S) \cap \partial M) \times I$ and further modify f with respect to these I-fibres so that $f(T) \cap \eta(\partial S) = (f(T) \cap \eta(S) \cap \partial M) \times I$. Once f has been thus modified by homotopy, it is called *normalized* with respect to $(M, \gamma) - S \to (\tilde{M}, \tilde{\gamma})$.

Lemma 6.2. *Given a branched immersion $f: T \to M$ normalized with respect to $(M, \gamma) - S \to (\tilde{M}, \tilde{\gamma})$, let $\tilde{f}: \tilde{T} \to \tilde{M}$ be defined by $\tilde{T} = T - f^{-1}(\operatorname{Int}(\eta(S)))$ and $\tilde{f} = f \mid \tilde{T}: (\tilde{T}, \partial \tilde{T}) \to (\tilde{M}, \partial \tilde{M})$. Then $I(f) = I(\tilde{f})$.*

Proof. Note that $f^{-1}(S)$ is a set of properly embedded arcs and circles in T, and cutting along $f^{-1}(S)$ gives the 2-manifold $\tilde{T}$. Cutting along a circle component does not change Euler characteristics and also corresponds to no new suture intersections since the new suture is in $\eta(\partial S)$, which cannot intersect the image of a circle component. Cutting along an arc α of T increases Euler characteristic by 1; each such arc α is mapped to an arc $f(\alpha)$ with endpoints in ∂S. After cutting M along S, each endpoint will correspond to a new suture intersection (either on S_+ or S_-). Hence $I(f) = I(\tilde{f})$. $\square$

Notation 6.3. Suppose (M, γ) is taut, and let

$$(M, \gamma) - S_1 \to (M_1, \gamma_1) - S_2 \to \cdots \to (M_{n-1}, \gamma_{n-1}) - S_n \to (M_n, \gamma_n)$$

be a hierarchy. Any element of Γ is admissibly homotopic to an element $f: T \to M$ of Γ with $f = f_0, f_1, \ldots, f_n$ all normalized, where $f_j = f: f^{-1}(M_j): f^{-1}(M_j) \to M_j$ for $j = 1, 2, \ldots, n$. Let Γ' be the subset of Γ consisting of all maps $f:(T, \partial T) \to (M, \partial M)$ such that

(i) $f = f_0, f_1, \ldots, f_n$ are all normalized,

(ii) T contains no sphere components,

(iii) if λ is a simple essential loop in T, then $f(\lambda)$ represents a nontrivial element in $\pi_1(M)$, and

(iv) $f(T) \cap \bar{C}$ contains no curves that self-intersect, i.e., each curve c_k is an embedding.

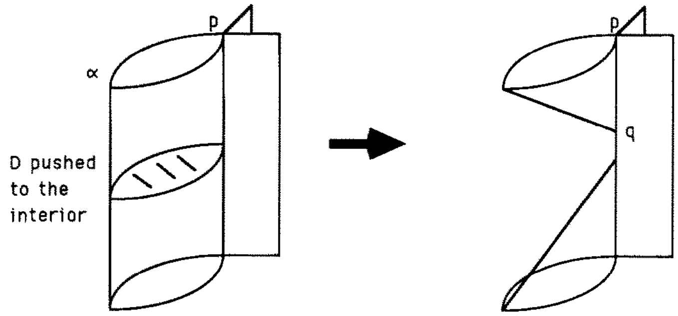

Fig. 8.

Lemma 6.4. *If M is closed, then any element $f: T \to M$ of Γ is homotopic to one satisfying* 6.3(iv).

Proof. The proof will be by induction on the number n of double points of $f(T) \cap C$. Clearly if $n = 0$, then no individual component c_k of $f(T) \cap \tilde{C}$ can have a double point.

Suppose then that p is a double point of some c_k, and let α be the singular subarc of c_k which begins and ends at p. Recall that $\tilde{C}$ is a disjoint union of spheres embedded in M; hence the loop α bounds a singular disk D on $\tilde{C}$. Push D slightly into the interior of $i(M_n)$ and use it to null-homotope the loop $\partial D \subset f(T)$. This has no effect on $f(T) \cap \tilde{C}$, but creates two new branch points on what was the double curve of $f(T)$ containing p. One of the arc remnants of that double curve runs from a new branch point q to the point p' in C corresponding to p. See the schematic diagram in Fig. 8.

Push the branch point q along the double curve past p'. This now changes $f(T) \cap C$—the effect is like a double curve sum near p. In particular, the double point p is removed. We have no idea what this does to the component(s) of $f(T) \cap \tilde{C}$ passing through the double point p'' on the other side of C from p'. In particular, if p'' lies on two different components of $f(T) \cap \tilde{C}$, these are joined together by the operation to become a single component, possibly with many new double points. But since we are inducting only on the number of double points of $f(T) \cap C$, this is completely irrelevant, and the proof is done by inductive assumption. □.

Definition 6.5. Let $f(T, \partial T) \to (M, \partial M)$ be in Γ'. If some component of T is a disk D with $f \mid \partial D$ an embedding and $f(\partial D) \cap \gamma = \emptyset$, then $f(\partial D)$ lies entirely inside $R_{\pm}(R_{\pm}$ is R_+ or $R_-)$. Since $R_{\pm}$ is incompressible, $f(\partial D)$ is null-homotopic in $R_{\pm}$ and so can be shrunk by an admissible homotopy as small as we wish, showing that $y(f \mid D) = \pm 1$. Call a disk component D of T with $f \mid \partial D$ an embedding, $f(\partial D) \subset R_{\pm}$ and $y(f \mid D) = +1$ a *deficiency disk*. Let Δ be the collection of all deficiency disks, and let $d(f)$ be the number of disks in Δ.

Lemma 6.6. *If $f:(T, \partial T) \to (M, \partial M)$ is in Γ', then $y(f) \leqslant \frac{1}{2}I(f) + 2d(f)$.*

Remark. If M and T are closed, $\Delta = \emptyset$ and $\nu = 0$ so we obtain $y(f) \leqslant -\chi(T)$ whenever $f \in \Gamma'$. Since M is irreducible, an arbitrary element $f: T \to M$ of Σ is homologous in $H_2(M)$ to map $f': T' \to M$ with T' having no sphere components, $f'(\lambda)$ nontrivial in $\pi_1(M)$ for each essential simple loop λ in T', and $\chi_-(T') \leqslant \chi_-(T)$. By Lemma 6.4, f' is homotopic to an element of Γ' and hence

$$y(f) = y(f') \leqslant \chi_-(T') \leqslant \chi_-(T).$$

Thus the proof of Lemma 6.6 completes the proof of property (iii) in Section 3.1.

Proof of Lemma 6.6.

Claim. *It suffices to prove the lemma in the case in which $d(f) = 0$.*

Proof. Let f' denote $f \mid T - \Delta$, and suppose $y(f') \leqslant \frac{1}{2}I(f')$. By definition, $y(f) = y(f') + d(f)$. Moreover, for any D in Δ, $I(f \mid D) = -2$ so $I(f) = I(f') - 2d(f)$. Combining these we get

$$y(f) - d(f) = y(f') \leqslant \tfrac{1}{2}I(f') = \tfrac{1}{2}I(f) + d(f),$$

proving the Claim.

The proof of the lemma will be by induction on the pair $(n, |T \cap S_1|)$ in lexicographical order, where n is the length of the hierarchy, S_1 is the first decomposing surface, and $|T \cap S_1|$ is the number of curves of $f(T) \cap S_1$. To begin the induction, suppose $n = 0$ so that (M, γ) is a disjoint union of 3-balls with one suture in each. Since $f \mid T \to M$ is in Γ', T is a union of disks. Suppose T has p components and let ξ_k be the number of points in $f(\delta_k) \cap i(\gamma)$, where $\delta_1, \ldots, \delta_p$ are the components of ∂T. By Notation 6.3(iv), each c_k is an embedding and since we are assuming $d(f) = 0$, each ρc_k either intersects δD^2 or travels counterclockwise. Hence, we may apply Note 4.3(iv) after the definition of y to conclude

$$y(f) \leqslant \tfrac{1}{2}\textstyle\sum(\xi_k - 2) = \nu/2 - p = \nu/2 - \chi(T) = \tfrac{1}{2}I(f).$$

This ends the case $n = 0$.

Inductively assume that $|T \cap S_1|$ has been minimized via admissible homotopies of T (which do not affect $y(f)$, $I(f)$, or $d(f)$). Let $\tilde{f}$, $\tilde{\Delta}$, etc. denote the relevant objects after decomposition along S_1.

We know that $y(\tilde{f}) = y(f)$ by definition, and $I(\tilde{f}) = I(f)$ by Lemma 6.2. By induction on n, $y(\tilde{f}) \leqslant \frac{1}{2}I(\tilde{f}) + 2d(\tilde{f})$, so if $d(\tilde{f}) = 0$, we are done. Suppose $d(\tilde{f}) \neq 0$ and let $D \in \tilde{\Delta}$. Then $\tilde{f}(\partial D)$ lies entirely in $\tilde{R}_+$, say, which is the union of pieces of R_+ and of a copy of S_1, denoted S_{1+}. Since $\tilde{R}_+$ is incompressible, $\tilde{f}(\partial D)$ is inessential in $\tilde{R}_+$.

Suppose $\tilde{f}(\partial D)$ lies entirely in S_{1+}. We have that $\tilde{f}(\partial D)$ bounds some disk E in $\tilde{R}_+$, and since $\tilde{f}(\partial D)$ lies entirely in S_{1+}, $\partial S_{1+} \cap E$ is a set of circles in E. Recall that no component of ∂S_1 bounds a disk in R_+ (cf. Remark 2.3). Thus an innermost circle of $\partial S_{1+} \cap E$ cannot bound a disk in R_+, so it is essential in R_+—and so it also cannot bound a disk in S_{1+}. Hence there can be no innermost circle, hence no circles of ∂S_{1+} in E at all, and so E is contained in S_{1+}. Since M is irreducible and f satisfies Notation 6.3(iii), $|T \cap S_1|$ can be reduced by an admissible homotopy of f (indeed one fixed on ∂T), contradicting the assumption that $|T \cap S_1|$ has been minimized.

Since $\Delta = \emptyset$, $\tilde{f}(\partial D)$ does not lie entirely in R_+. Thus $\tilde{f}(\partial D)$ is the union of arcs in $(S_1)_+$ and arcs in R_+, and $2|\partial D \cap S_1| = |\partial D \cap \partial S_1|$. Since $\tilde{f}(\partial D)$ is inessential in $\tilde{R}_+$, some of these arcs are inessential. But if any such arc in R_+ is inessential, then $|\partial D \cap \partial S_1|$, hence $|\partial D \cap S_1|$ could be reduced by an admissible homotopy of f. Hence there is a properly embedded arc α of $f(T)$ such that α lies in S_1 and is homotopic rel endpoints to a subarc β of ∂S_1 lying entirely in R_+.

There is an admissible homotopy of T which makes α and β so small that they are disjoint from $\eta(S_2), \eta(S_3), \ldots, \eta(S_n)$. Let $f' : T' \to M$ be the map obtained from f by ∂-compressing T along the subdisk of S_1 bounded by $\alpha \cup \beta$. Note that the curves $f'(T') \cap \tilde{C}$ are isotopic to the curves $f(T) \cap \tilde{C}$, so $f' \in \Gamma'$. Since $|T' \cap S_1| < |T \cap S_1|$ we have, by the inductive assumption, $y(f') \leqslant \frac{1}{2}I(f') + 2d(f')$. Moreover, $y(f') = y(f) \pm 1$, for the effect of the compression is to replace the copy of the arc α which lies in $g'(D^2 \times \{-1\})$ with a copy of β lying in $g'(D^2 \times \{1\})$, where g' is the restriction of g to one copy of $D^2 \times I$.

If f' contains any deficiency disks, then they must contain one or both of the copies of the subdisk of S_1 with boundary $\alpha \cup \beta$. Let D_- and D_+ be the copies in T' of the subdisk of S_1 with boundary $\alpha \cup \beta$, chosen so that in a regular neighborhood of S_1, $f'(D_+)$ is on the same side of S_1 as $(S_1)_+$ and $f'(D_-)$ is on the same side of S_1 as $(S_1)_-$, where $(S_1)_-$ is the parallel copy of S_1 that becomes part of $\tilde{R}_-$ after decomposition.

If f' has two distinct deficiency disks then these must be the result of ∂-compressing a deficiency disk of f, contradicting $d(f) = 0$.

Suppose f' has one deficiency disk E containing D_- but not D_+. Then $f'(\partial E)$ must rotate clockwise in M_n so the orientation of ∂T is as shown in Figs. 9(a) and (b). Hence, in this case $y(f) = y(f') - 1$.

Similarly, if f' has one deficiency disk E containing D_+ but not D_-, then since $f'(\partial E)$ rotates clockwise in M_n, the orientation of ∂T is as shown in Fig. 10. Again we have $y(f) = y(f') - 1$.

If f' has one deficiency disk containing both D_- and D_+, then one can also similarly show that $y(f) = y(f') - 1$.

Note that $\chi(T') = \chi(\mathrm{T}) + 1$ so $I(f') = I(f) - 2$. Thus, if f' has a deficiency disk, then $d(f') = 1$ so

$$y(f) = y(f') - 1 \leqslant \tfrac{1}{2}I(f') + 2d(f') - 1 = \tfrac{1}{2}I(f).$$

If f' has no deficiency disks, then $d(f') = 0$ so

$$y(f) = y(f') \pm 1 \leqslant y(f') + 1 \leqslant \tfrac{1}{2}I(f') + 1 = \tfrac{1}{2}I(f). \quad \square$$

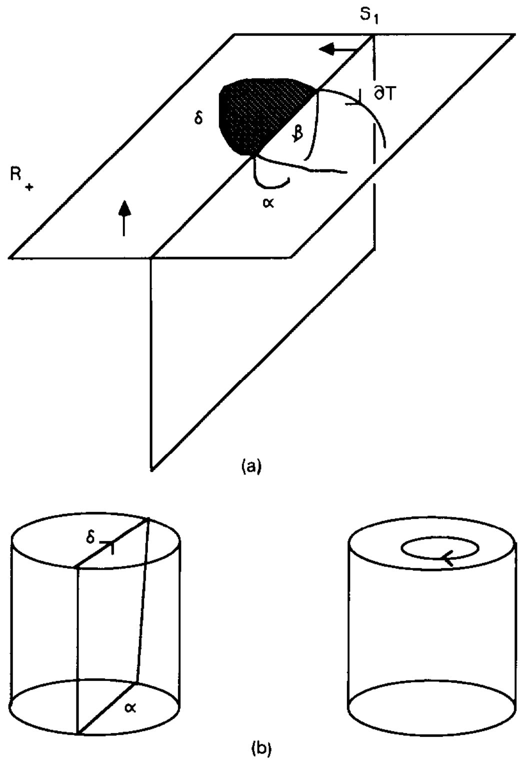

Fig. 9. (a) After cutting along S^1, suture appears on the left-hand side of this picture. The intersection of the surface with this suture component before and after compression is illustrated in Fig. (b). (b) The right image in M_n before compression is shown on the left, and the image of $f'(\partial E)$ in M_n is shown on the right.

7. Proof of property (iv) in Section 3.1: If M is closed, then given a nonzero $\tau \in H_2(M)$, there exists a hierarchy and an embedding $g: T \to M$ with $y(g) = \chi_-(T)$ and $g_*[T] = \tau$

By Theorem 2.2, there exists a taut sutured manifold hierarchy

$$(M, \emptyset) - T \to (M_1, \gamma_1) - S_2 \to \cdots \to (M_{n-1}, \gamma_{n-1}) - S_n \to (M_n, \gamma_n)$$

with $[T] = \tau$. Let $g: T \to M$ be inclusion. Then an explicit calculation will show $y(g) = \chi_-(T)$; to carry out the calculation a carefully chosen collar on T will be constructed.

Definition 7.1 (of N_j, T_j, and Q_j for $j = 1, 2, \ldots, n$). Let N_1 be a regular neighborhood in M_1 of the R_+ of M_1. Then ∂N_1 consists of two components, each of which, when considered as a subset of M, is parallel to T. Let T_1 be the boundary component of N_1 which is R_+ for M_1, and orient T_1 as R_+ is oriented. Let Q_1 be

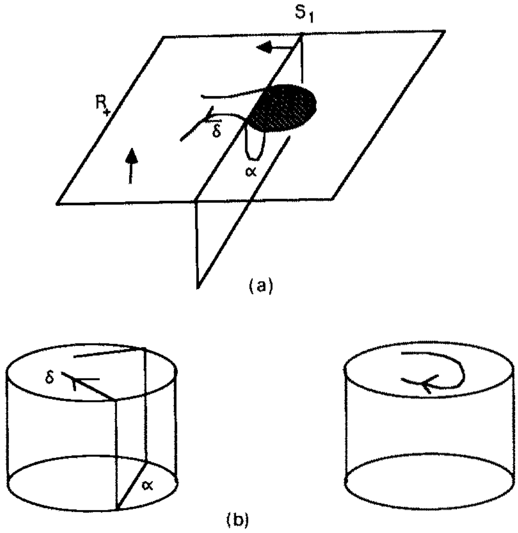

Fig. 10. (a) After cutting along S_1, again suture appears on the left-hand side of this picture. The intersection of the surface with this suture component before and after compression is illustrated in Fig. (b). (b) The image in M_n before compression is shown on the left, and the image in M_n after compression is shown on the right.

the other boundary component of N_1, oriented so that $\partial N_1 = T_1 \cup (-Q_1)$. Roughly, N_j, T_j, and Q_j are the results of cutting N_1, T_1, and Q_1 (respectively) along $S_2, \ldots, S_j$; however, we want to also preserve a correspondence between T_j and Q_j.

Fix a fibering $N_1 = T_1 \times [0, 1]$ with T_1 identified with $T_1 \times \{0\}$ and Q_1 identified with $T_1 \times \{1\}$. Perform an isotopy on S_2 so that

$$S_2 \cap N_1 = S_2 \cap (T_1 \times [0, 1]) = (S_2 \cap T_1) \times [0, 1].$$

Now, $M_2 = M_1 - \mathrm{Int}(\eta(S_2))$ for some regular neighborhood $\eta(S_2)$ of S_2 in M_1; choose $\eta(S_2)$ so that

$$\eta(S_2) \cap N_1 = \eta(S_2) \cap (T_1 \times [0, 1]) = (\eta(S_2) \cap T_1) \times [0, 1].$$

Define $N_2 = N_1 - \mathrm{Int}(\eta(S_2))$. Then N_2 inherits a fibering $N_2 = T_2 \times [0, 1]$, where $T_2 = T_1 - \mathrm{Int}(\eta(S_2))$. Put $Q_2 = Q_1 - \mathrm{Int}(\eta(S_2))$ and note that $Q_2 = T_2 \times \{1\} \subset T_2 \times [0, 1]$. Continue in this way by performing isotopies on S_j and $\eta(S_j)$ so that they respect the $[0, 1]$-fibers of N_{j-1} and then define

$$T_j = T_{j-1} - \mathrm{Int}(\eta(S_j)),$$

$$Q_j = Q_{j-1} - \mathrm{Int}(\eta(S_j)),$$

and

$$N_j = N_{j-1} - \mathrm{Int}(\eta(S_j)),$$

for $j = 2, 3, \ldots, n$.

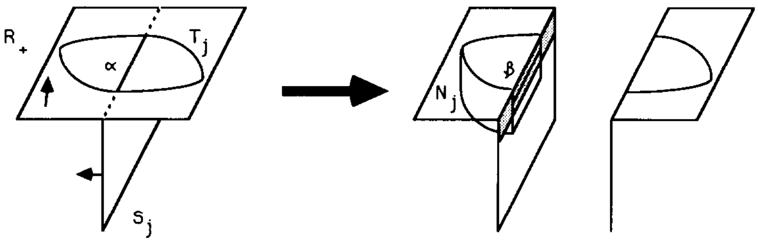

Fig. 11.

Claim 7.2. *For $j = 1, 2, \ldots, n$, we may assume that*

(i) *for each $x \in T_j$, the $[0, 1]$-fiber $\{x\} \times [0, 1]$ in $N_j = T_j \times [0, 1]$ either misses γ_j or meets γ_j in a single point or a single arc, and*

(ii) *for each $\gamma \in \gamma_j$, the I-fiber $\{y\} \times I$ and $A(\gamma_j) = \gamma_j \times I$ either does not meet N_j or is entirely contained in N_j (possibly transverse to its fibers).*

Proof. Since T is closed, $\gamma_1 = \emptyset$ so the claim is vacuously true for $j = 1$. Suppose the claim holds for $j = k - 1$.

In Definition 7.1, we already performed an isotopy on S_k so that S_k meets N_{k-1} in $[0, 1]$-fibers. Since S_k is transverse to γ_{k-1} and since we are assuming (i) holds for $j = k - 1$, $(\{x\} \times [0, 1]) \cup \gamma_{k-1}$ is either empty or one point whenever $x \in T_{k-1} \cap S_k$. By shrinking $\eta(S_k)$ if necessary, we may also assume $(\{x\} \times [0, 1] \cap \gamma_{k-1}$ is either empty or a single point for $x \in T_{k-1} \cap \eta(S_k)$.

Now consider how γ_k is obtained from γ_{k-1} and S_k. It may be helpful to refer to Fig. 2.

Case 1: *Suppose that α is an arc in $T_{k-1} \cap S_k$ with $\alpha \times [0, 1]$ disjoint from γ_{k-1}.* Then α corresponds to an arc β of γ_k parallel to α. In this case β is transverse to the $[0, 1]$-fibers of N_k and so (i) is satisfied for $j = k$ near α. By shrinking $A(\gamma_k)$ (if necessary) and aligning the I-fibers of $A(\gamma_k) = \gamma_k \times I$ suitably, we may require $\beta \times I \subset N_k$, thus satisfying (ii) near α. See Fig. 11.

Case 2: *Suppose, on the other hand, that $x \in T_{k-1} \cap S_k$ and $(\{x\} \times [0, 1]) \cap \gamma_{k-1}$ consists of one point*. Here $\{x\} \times [0, 1]$ will correspond to an arc ∂ of γ_k parallel to the $[0, 1]$-fibers of N_j. Align the I-fibers of $A(\gamma_k) = \gamma_k \times I$ so that (ii) is also satisfied. See Fig. 12. □

Proof that $y(g) = \chi_-(T)$. Since T is incompressible and M is irreducible, Q_n contains no essential closed curves and has no spherical components; hence Q_n is a disjoint union of disks. Moreover, the product structure on $N_n = T_n \times [0, 1]$ between $T_n \times \{0\}$ and $Q_n = T_n \times \{1\}$ determines a product structure on ∂M_n between ∂Q_n and ∂T_n. By Claim 7.2, each $[0, 1]$-fiber of N_n meets γ_n, if at all, in a single point or a single arc. It follows that corresponding curves in ∂Q_n and ∂T_n are concentric with γ_n in the 2-sphere component of ∂M_n on which they both lie. In particular, if δ is a component of ∂Q_n then $\rho\delta$ moves counterclockwise if and only

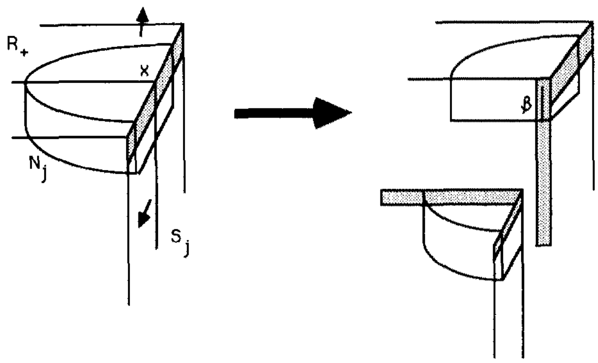

Fig. 12.

if the corresponding curve of ∂T_n does. But $T_n \subset R_+$ and the orientation on T_n agrees with the orientation on R_+, so each curve of ∂T_n—and hence $\rho\delta$ for each component δ of ∂Q_n—moves counterclockwise.

Let i: $Q_1 \to M$ be inclusion; note that $i \in \Gamma$ and so $y(g) = y(i)$ by definition. Also,

$$\chi_-(T) = -\chi(Q_1) = \tfrac{1}{2}I(i) = \tfrac{1}{2}I(i \mid Q_n) = \tfrac{1}{2}\sum(\xi_k - 2),$$

where ξ_k is the number of intersections of the kth component of ∂Q_n with γ_n and the sum is over all components of ∂Q_n. By note 4.3(iii),

$$\tfrac{1}{2}\sum(\xi_k - 2) = y(g). \quad \square$$

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